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Quark Theory with Internal Coordinates

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Abstract

The free-particle Dirac wave function $\psi(x)$ has been generalized to $\psi(x)\xi^a(z)$. Here, z denotes a set of three complex coordinates, called internal coordinates, in an abstract complex three-dimensional space, called internal space; a runs from 1 to 3; and $\xi^a(z)$ is assumed to contain a representation of the state of a quark triplet. The mass in the free-particle Dirac equation is replaced by a second-order operator $\partial_a{}^b$ operating on $\xi^a(z)$. The Dirac equation so modified is assumed to include a description of a free quark triplet. Subsequently, symmetry-preserving interactions between two quark triplets are introduced. Two SU_3 symmetry-breaking interactions, one transforming like the eighth component of an SU_3 octet vector and the other like the SU_3 charge operator, are also introduced. A similarly generalized Bethe-Salpeter equation in the ladder approximation was obtained. This equation has been treated in greater detail in an accompanying paper in which the Gell-Mann–Okubo formula for psuedoscalar mesons was derived with the coefficients determined by given relations. Then, spherical coordinates and corresponding spherical harmonics in the internal space are introduced. Finally, the equation for a one-quark system is briefly treated.

1. *Introduction*

Quantum mechanics, formulated about half a century ago, has indeed served us very well in the field of atomic physics. Quantum field theory, formulated shortly afterwards, together with the renormalization technique introduced about a quarter a century ago has been spectacularly confirmed by experiments in the case of electromagnetic interactions, at the present time down to 10^{-15} cm, a distance considerably smaller then the length scale of a nucleus.

When applied to nuclear phenomena and strong interactions among the so-called elementary particles, quantum theory has proved to be far less satisfactory in accounting for experiments, both qualitatively and quantitatively. There are several reasons for this. In the first place, we do not have an accurate or even adequate representation of the strong interaction in spite of a great wealth of experimental data, as we do in the case of the electromagnetic interaction. Thus, although we know the strong interaction phenomenologically,

qualitatively, and semiquantitatively, there are features of strong interactions that cannot be naturally fitted into the quantum theoretical description of today. Secondly, the strong interaction strength is greater than unity, and the perturbational approach, which is possible in quantum electrodynamics and is essential in bringing it into contact with experiments, in principle fails. For the above reasons, the dispersion-relations approach of quantum theory to strong interactions was devised to complement the field theoretical approach. In spite of its partial success, the dispersion-relations approach is to be considered as an auxiliary theory in our search for a more adequate description of the strong interaction if our belief is that such a description is to take the form of a system of partial differential, or differential integral, equations. The last belief is based upon our experiences in all of the different basic branches of physics. Thirdly, particle masses are to be introduced as parameters into quantum theory. This is practically manageable as long as the number of different masses is small. In an elementary particle interaction, however, a great number of particles with different masses can be produced. To account for such interactions with quantum theory, the large number of different masses causes the theory to assume a rather clumsy appearance.

For these reasons, at least, it is evident that quantum theory must be suitably modified or wholly replaced by another theory if we are to account for strong interactions more satisfactorily. The purpose of this paper is to present a possible such theory. In the search for such a more satisfactory theory, one may be guided by the above discussion and look for a system of partial differential, or differential integral, equations. The great success of quantum electrodynamics further suggests that at least some basic features of quantum field theory have fundamental importance and hold as well for interactions other than the electromagnetic one. This is confirmed by the partial success of quantum field theory applied to strong interactions. Therefore, quantum theory will be used as a starting point and be suitably modified or generalized.

Another line of approach to account for our experiences in nuclear and strong interaction phenomena began in the 1930's when the isospin formalism, associated with the mathematical group SU_2, was proposed and turned out to be highly successful in classifying these phenomena. Generalization of the isospin concept to include strangeness or hypercharge and the corresponding generalization of SU_2 to SU_3 took place during the 1950's and early 1960's. Based on these generalizations, Gell-Mann (1962) and Néeman (1961) showed that essentially all the hadrons then known could be fitted into an SU_3 classification scheme. Moreover, it was proposed that the masses of these particles were to be considered as eigenvalues of a mass operator operating on a set of functions representing an irreducible SU_3 multiplet. In particular, the mass operator was assumed to consist of an SU_3 singlet term and a term transforming like the eighth component of an SU_3 octet vector (Okubo, 1962). In this manner, the Gell-Mann–Okubo formula relating the masses of particles within a given SU_3 multiplet was deduced and agreed well with experimental data under certain assumptions. A similar set of relations, concerning the electromagnetic splittings in SU_3 multiplets only, was also derived

(Coleman and Glashow, 1961) and likewise agreed well
data. Along this line, Gell-Mann (1964) and Zweig (19
tially all the hadrons then known could be considered to
combinations of a fundamental triplet, called quarks, much li
mental doublet, proton and neutron, which make up atomic nuc

The SU_3 and quark formalism is so practical and powerful that it
that it has become of fundamental importance to strong-interaction phy
By its very nature, however, the formalism, apart from spin, does not conta
any aspect of space-time mechanics of strong interactions. Since the experimental data are functions of space-time, the formalism alone is obviously insufficient in accounting for hadron interactions.

In the search for a more satisfactory theory for strong interactions, therefore, it appears natural to try to combine quantum theory and the SU_3 and quark formalism in a suitable way. The above suggestion that quantum theory is to provide a starting point and is to be suitably generalized can now be made more specific; the generalization consists of including the SU_3 and quark formalism or its equivalent in a suitable way. In particular, the suggestion leading to the Gell-Mann–Okubo formula, that masses are eigenvalues of a mass operator, has been substantially followed. Further, use has been made of the hypothesis that quarks, formally represented by a triplet ξ^a, where a runs from 1 to 3, or included in such a formal representation for the time being, are the fundamental constituents from which all hadrons are built.

Since quarks have spin $\frac{1}{2}$, Dirac's equation for a free particle will be used as the starting point. The Dirac wave function is associated with a triplet ξ^a representing the quarks, and the mass term in the Dirac equation is replaced by a suitable operator m_{op} operating on ξ^a. In this process, an abstract three-dimensional complex space, called internal space and denoted by M_3, together with a set of three complex coordinates, called internal coordinates and denoted by z, are introduced. ξ^a then takes the meaning $\xi^a(z)$ and m_{op} takes the form of a second-order tensor operator $\partial_a{}^b$. These steps are carried out in Section 2 and the resulting equation is assumed to hold for a free-quark triplet or simply a free quark.

In Section 3, a quark is assumed to interact with another quark with known state. Interactions in space-time as well as in the internal space are introduced. These interactions, however, preserve both the Lorentz and the U_3 invariance of the equations. In Section 4, interactions breaking the SU_3 invariance of the equations are introduced. In particular, an interaction term in the internal space transforming like the eighth component of an SU_3 octet vector is introduced following the line of Okubo (1962).

In Section 5, the above generalizations are extended to the Bethe-Salpeter equation in the ladder approximation. The Bethe-Salpeter equation thus generalized for a quark–antiquark pair may account for mesons. This equation is treated in greater detail in an accompanying paper (Hoh, 1975) in which the Gell-Mann–Okubo formula for psuedoscalar mesons is derived with the coefficients determined by given relations.

In Section 6, spherical coordinates and corresponding spherical harmonics

in the internal space are introduced following the work of Bég and Ruegg (1965).

In Section 7, the internal part of the free-quark equation is solved for the case of a single free quark. Solution is also given when an SU_3 singlet interaction function arising from a point source fixed in the internal space is included.

2. *The Free-Quark Equation*

The free-particle Dirac equation reads

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi(x) = 0 \tag{2.1}$$

Since one may anticipate that the triplet ξ^a will be involved in a generalization of (2.1), (2.1) is rewritten in the spinor form of van der Waerden (1929):

$$i\partial_{\sigma}{}^{\nu}\eta^{\sigma}(x) - m\chi^{\nu}(x) = 0, \quad i\partial_{\nu}^{\dot{\tau}}\chi^{\nu}(x) + m\eta^{\dot{\tau}}(x) = 0 \tag{2.2}$$

in order to facilitate a comparison between $\psi(x)$ and ξ^a. $\eta^{\dot{\sigma}}(x)$ and $\chi^{\nu}(x)$, on the one hand, and ξ^a, on the other, are now on equal footing: $\dot{\sigma} = 1, 2$ or $\nu = 1, 2$ refer to the spin-up and spin-down states of a particle and $a = 1, 2$ refer to the isospin-up and isospin-down states of the same particle neglecting electromagnetic interactions. $a = 3$ refers to a state having a nonvanishing strangeness. From van der Waerden (1929) and Laporte and Uhlenbeck (1931) we can obtain

$$\eta^{\dot{1}} = \psi_2 + \psi_4, \quad \eta^{\dot{2}} = -\psi_1 - \psi_3, \quad \chi^1 = \psi_1 - \psi_3, \quad \chi^2 = \psi_2 - \psi_4 \tag{2.3}$$

$$\begin{pmatrix} x_{1\dot{1}} & x_{2\dot{1}} \\ x_{1\dot{2}} & x_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad \begin{pmatrix} \partial_{1\dot{1}} & \partial_{2\dot{1}} \\ \partial_{1\dot{1}} & \partial_{2\dot{2}} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial}{\partial x_3} - \dfrac{\partial}{\partial x_0} - \dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_1} + i\dfrac{\partial}{\partial x_2} - \dfrac{\partial}{\partial x_3} - \dfrac{\partial}{\partial x_0} \end{pmatrix} \tag{2.4}$$

$$\Box = -\tfrac{1}{2}\partial_{\dot{\alpha}\beta}\partial^{\beta\dot{\alpha}} \tag{2.5}$$

χ^{ν} transforms like a contravariant spinor or doublet function and is analogous to ξ^a, which transforms as a contravariant triplet function. The covariant spinor χ_{ν} is defined by $\chi_{\nu} = \epsilon_{\nu\sigma}\chi^{\sigma}$, where

$$\epsilon_{\nu\sigma} = -\epsilon_{\sigma\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.6}$$

is a totally antisymmetric tensor. Similarly, $\eta^{\dot{\sigma}}$ transforms like the complex conjugate of a contravariant spinor and $\eta_{\dot{\sigma}}$ like that of a covariant spinor. $\eta^{\dot{\sigma}} = \epsilon^{\dot{\sigma}\dot{\nu}}\eta_{\dot{\nu}}$, $\eta_{\dot{\sigma}} = \epsilon_{\dot{\sigma}\dot{\nu}}\eta^{\dot{\nu}}$, $\epsilon^{\dot{\sigma}\dot{\nu}}$ has the same form as $\epsilon^{\sigma\nu}$, and $\epsilon_{\dot{\sigma}\dot{\nu}}$ as $\epsilon_{\sigma\nu}$ in (2.6). The physical quantities $x_{\alpha\dot{\beta}}$ and physical operators $\partial_{\alpha\dot{\beta}}$ transform like a mixed tensor of second rank. Entities like x_{α}, $x_{\dot{\beta}}$, ∂_{α}, and $\partial_{\dot{\beta}}$ do not exist in the physical world.

The generalization of (2.2) to account for a free quark makes use of the fact that a triplet ξ^a has been assigned to represent quarks and of the earlier suggestion that m is the eigenvalue of an operator m_{op} operating on a set of an irreducible SU_3 multiplet. In such a generalization, ξ^a is obviously to be associated with χ^ν or η^σ in a multiplicative manner. It is natural to assume that ξ^a can represent an irreducible SU_3 multiplet, namely, an SU_3 triplet; m_{op} then obviously operates on ξ^a. The formal form of m_{op} is chosen according to the following discussion. In classical mechanics, both the momenta p_μ, or in spinor form p_ν^τ, and the mass m are observables. In going over to quantum mechanics, the observable p_ν^τ was replaced by the operator $i\partial_\nu^\tau$, which operates on a spinor function χ^ν introduced in this connection. The observable m remained unchanged. Presently, m is also to be replaced by an operator m_{op} operating on the triplet ξ^a introduced. Now ξ^a is on equal footing with χ^ν, as discussed after (2.2). Therefore, m_{op} is assumed to take the formal mathematical form const $\partial_a^{\dot b}$ by analogy with the form $i\partial_\nu^{\dot\tau}$. Putting const $= -1$, (2.2) is thus formally generalized to

$$\begin{aligned} i\partial_{\dot\sigma}\,{}^{\nu}\eta^{\dot\sigma}(x)\xi^c + \partial_b^{\,c}\xi^{\dot b}\chi^\nu(x) &= 0 \\ i\partial_\nu^{\dot\tau}\chi^\nu(x)\xi^{\dot b} - \partial_a^{\,\dot b}\xi^a\eta^{\dot\tau}(x) &= 0 \end{aligned} \tag{2.7}$$

which possess a formal symmetry between the spinor Greek indices and the triplet Latin indices.

The existence of $\eta^{\dot\tau}$ transforming like the complex conjugate of χ^τ is associated with the existence of positive and negative energy components of $\psi(x)$. The latter existence depends upon the existence of energy which together with momentum form a Lorentz metric. If there were only positive energy components, like those appearing in the nonrelativistic case in which the metric is Euclidean, it would not be necessary to carry the dotted Greek indices. From our experience in working with the quark contravariant triplet function ξ^a, we have not encountered any aspect analogous to the above-mentioned positive and negative energy aspects. The antiquark triplet function can be represented by the covariant triplet ξ_a. Also, as is shown later in (2.15), the metric to be associated with ξ^a is not Lorentian but Euclidean. Therefore, the dots on the index b in (2.7) can be and are removed and (2.7) becomes

$$\begin{aligned} i\partial_{\dot\sigma}\,{}^{\nu}\eta^{\dot\sigma}(x)\xi^c + \partial_b^{\,c}\xi^b\chi^\nu(x) &= 0 \\ i\partial_\nu^{\dot\tau}\chi(x)\xi^b - \partial_a^{\,b}\xi^a\eta^{\dot\tau}(x) &= 0 \end{aligned} \tag{2.8}$$

$\partial_b^{\,a}\xi^b$ is now interpreted as

$$\partial_b^{\,a}\xi^b = \frac{\partial^2}{\partial z_a \partial z^b}\xi^b(z^1, z^2, z^3, z_1, z_2, z_3) \tag{2.9}$$

where $(z^1, z^2, z^3) \equiv z^a$ are three complex independent variables spanning an abstract complex three-dimensional space M_3. Aspects of such a space have been discussed earlier by Bég and Ruegg (1965) and Tait (1972). $\xi^a(z^b, z_b)$ is assumed to transform as z^a or as a contravariant vector or triplet in M_3.

The covariant space-time spinor χ^{ν} is defined in terms of the contravariant spinor χ^{σ} by means of the antisymmetric tensor $\epsilon_{\nu\sigma}$ in (2.6). The tensor, corresponding to $\epsilon_{\nu\sigma}$, in M_3 is the totally antisymmetric tensor ϵ_{abc} where a, b, and c each runs from 1 to 3. There is no ϵ_{ab} that can be used to define a covariant vector $z_a = (z_1, z_2, z_3)$ in terms of z^b in M_3. One is therefore free to define z_a as the Hermitian conjugate of z^a:

$$z_a = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = (z^a)^\dagger = (z^1, z^2, z^3)^\dagger = \begin{pmatrix} z^{1*} \\ z^{2*} \\ z^{3*} \end{pmatrix} \tag{2.10}$$

$\xi_a(z^1, z^2, z^3, z_1, z_2, z_3)$, representing the antiquark triplet, is similarly defined as $(\xi^a)^\dagger$ and transforms like a covariant vector or z_a in M_3.

Consider a unitary transformation in M_3: $z'_a = T_a{}^b z_b$. Its Hermitian conjugate is $z'^\dagger_a = z^{a'} = z^b T_b{}^{a*}$. Unitarity requires that

$$z^{a'} z'_a = z^b T_b{}^{a*} T_a{}^c z_c = z^b z_b \tag{2.11}$$

so that $T_b{}^{a*} T_a{}^c = \delta_b{}^c$. Using these relations, one can show that

$$\partial^{a'} \equiv \frac{\partial}{\partial z'_a} = \frac{\partial z_b}{\partial z'_a} \frac{\partial}{\partial z_b} = T_b{}^{a*} \frac{\partial}{\partial z_b} \equiv T_b{}^{a*} \partial^b \tag{2.12}$$

and thus transforms like a contravariant vector. Similarly

$$\partial'_a \equiv \frac{\partial}{\partial z^{a'}} = \frac{\partial z_b}{\partial z'_a} \frac{\partial}{\partial z_b} = T_a{}^b \frac{\partial}{\partial z^b} \equiv T_a{}^b \partial_b \tag{2.13}$$

and transforms like a covariant vector. The mass operator $-\partial_b{}^a \equiv -\partial^2/\partial z_a \partial z^b$ transforms like a mixed tensor of second rank in M_3 and is analogous to the energy momentum operator $i\partial_\sigma{}^\nu$ in (2.8), which transforms like a mixed tensor of second rank in spinor space. The contracted mass operator $-\partial_a{}^a$ and contracted mass operator squared $\partial_b{}^a \partial_a{}^b$ transform like scalars in M_3; the latter is analogous to the d'Alembertian operator $\Box$ in (2.5). Writing

$$z^1 = y_1 + iy_2, \quad z^2 = y_3 + iy_4, \quad z^3 = y_5 + iy_6 \tag{2.14}$$

where the y's are real quantities, the invariant quantity $z^b z_b$ in (2.11) becomes

$$z^b z_b = \sum_{i=1}^{6} y_i^2 \tag{2.15}$$

showing that the metric in M_3 is Euclidean.

With the interpretation (2.9) and the discussions and definitions that follow, (2.8) becomes well defined. The spinor free-quark equations of (2.8) can be combined to produce a bispinor free-quark equation that is the corresponding generalization of (2.1), namely,

$$i\gamma^\mu \partial_\mu \psi(x) \xi^a(z) + \partial_b{}^a \psi(x) \xi^b(z) = 0 \tag{2.16}$$

Here, z denotes $(z^1, z^2, z^3, z_1, z_2, z_3)$ and will be referred to as internal coordinates.

Similarly, M_3 can be called as internal space and $\xi^a(z)$ an internal quark function One can now say that a principle guiding the quantum mechanical equations (2.2) to the generalized equations including internal coordinates, (2.7), is that the space-time part and the internal part of the generalized equations (2.7) are put on equal footing. Equation (2.16) can be separated to give a space-time part in the form of a Dirac equation

$$i\gamma^\mu \partial_\mu \psi(x) - m_{g0}\psi(x) = 0 \tag{2.17}$$

and an internal part in the following form

$$\partial_b{}^a \xi^b(z) + m_{g0}\xi^a(z) = 0 \tag{2.18}$$

Where m_{g0} is a mass separation constant between the space-time part and the internal part of the free-quark equation (2.16).

The transition from the Dirac equation for a free particle, (2.1), to the free-quark equation, (2.16), can be achieved by following the formal prescription: Multiply (2.1) by an internal quark function $\xi^b(z)$ from the right, replace m by $-\partial_b{}^a$, and multiply the term in the parentheses that does not involve m by $\delta_b{}^a$. This prescription is similar to that taking classical mechanics to quantum mechanics. The classical Hamiltonian for a free particle is

$$p_\mu p^\mu - m^2 = 0 \tag{2.19}$$

The prescription for going over to quantum mechanics is the following: Multiply (2.19) from the right by a wave function $\varphi(x)$ and replace p_μ by $i\partial/\partial x^\mu$and p^μ by $i\partial/\partial x_\mu$. Carrying out these steps one obtains the Klein-Gordon equation for the same particle

$$(\Box - m^2)\varphi(x) = 0 \tag{2.20}$$

If one considers the particle to be a psuedoscalar meson, one can further generalize (2.20) by following a prescription similar to that applied to the free-particle Dirac equation (2.1) mentioned between (2.18) and (2.19): Multiply (2.20) by an SU_3 singlet internal function $\tau(z)$ from the right and replace m^2 by $\partial_d{}^c \partial_c{}^d$. Carrying out these two steps one obtains

$$(\Box - \diamondsuit)\varphi(x)\tau(z) = (-\tfrac{1}{2}\partial_{\alpha\dot{\beta}}\partial^{\dot{\beta}\alpha} - \partial_d{}^c\partial_c{}^d)\varphi(x)\tau(z) = 0 \tag{2.21}$$

where

$$\diamondsuit \equiv \partial_d{}^c \partial_c{}^d = \partial_c{}^c \partial_d{}^d \tag{2.22}$$

and (2.5) has been used.

When going from the classical description (2.19) to the quantum mechanical description (2.20), the observables p_μ were replaced by operators but the observable m was not. When going from the quantum mechanical description (2.20) to the generalized description including internal coordinates, (2.21), the observable m was also replaced by an operator. All the observable quantities in the classical Hamiltonian (2.19), p_μ and m, have now been replaced by operators and in that sense been put on equal footing. Equation

(2.21) exhibits a symmetry between its space-time part and its internal part just like (2.16) does. Like (2.16), (2.21) can be separated to give

$$(\Box - m_m^2)\varphi(x) = 0 \tag{2.23}$$

$$(\diamondsuit - m_m^2)\tau(z) = 0 \tag{2.24}$$

which are analogous to (2.17) and (2.18), respectively. m_m^2 is the mass separation constant between the space-time part and the internal part of the free meson equation (2.21).

3. *Symmetry-Preserving Interactions*

Consider a fermion with mass m and wave function $\psi(x)$ interacting with another fermion with a known wave function $\chi(x)$ through the exchange of a, say, psuedoscalar particle with mass m_{p0}. The appropriate equations are

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = i\gamma_5 U_p(x)\psi(x) \tag{3.1}$$

$$(\Box - m_{p0}^2)U_p(x) = \mu_p \bar{\chi}(x)\gamma_5 \chi(x) \tag{3.2}$$

where μ_p is a psuedoscalar interaction parameter. These two equations are generalizations of (2.1) and (2.20), respectively.

The generalization of these equations to include internal coordinates and functions begins by following the prescriptions given between (2.18) and (2.19) for (3.1) and those between (2.20) and (2.21) for (3.2). Multiplying (3.1) by $\xi^b(z)$ from the right, replacing m by $-\partial_b{}^a$, and multiplying the terms not containing m by $\delta_b{}^a$, one has

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b{}^a \psi(x)\xi^b(z) = i\gamma_5 U_p(x)\psi(x)\xi^a(z) \tag{3.3}$$

Multiplying (3.2) by $\tau(z)$ from the right and replacing m_{p0}^2 by $\diamondsuit$, one obtains

$$(\Box - \diamondsuit)U_p(x)\tau(z) = \mu_p \bar{\chi}(x)\gamma_5 \chi(x)\tau(z) \tag{3.4}$$

The left side of (3.3), just like that of (2.16), and the left side of (3.4), similar to (2.21), both possess symmetry between space-time and internal parts. The right sides of these equations, one representing an interaction and the other a source function in space-time, do not possess such a symmetry. The next step in the generalization consists of providing such a symmetry by completing the right sides of (3.3) and (3.4) in such a way that these equations become

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b{}^a \psi(x)\xi^b(z) = i\gamma_5 U_p(x)\psi(x)\xi^a(z) + \tau(z)\psi(x)\xi^a(z) \tag{3.5}$$

$$(\Box - \diamondsuit)U_p(x)\tau(z) = \mu_p \bar{\chi}(x)\gamma_5 \chi(x)\tau(z) - \mu_0 \zeta_c(z)\xi^c(z)U_p(x) \tag{3.6}$$

where μ_0 is an SU_3 singlet interaction parameter. The space-time wave function $\chi(x)$, associated with one of the fermions, has been generalized to the quark triplet wave function or simply quark function $\chi(x)\zeta^a(z)$ just as $\psi(x)$ was generalized to $\psi(x)\xi^a(z)$ and as the psuedoscalar interaction function $U_p(x)$ was generalized to the psuedoscalar SU_3 singlet interaction function $U_p(x)\tau(z)$.

As with $\chi(x)$, $\zeta^a(z)$ is assumed to be known. Equation (3.5) can be separated to give

$$i\gamma^\mu \partial_\mu \psi(x) - m_g \psi(x) = i\gamma_5 U_p(x)\psi(x) \tag{3.7}$$

$$\partial_b{}^a \xi^b(z) + m_g \xi^a(z) = \tau(z)\xi^a(z) \tag{3.8}$$

and (3.6) to give

$$(\Box - m_p{}^2)U_p(x) = \mu_p \bar{\chi}(x)\gamma_5 \chi(x) \tag{3.9}$$

$$(\diamondsuit - m_p{}^2)\tau(z) = \mu_0 \zeta_c(z)\zeta^c(z) \tag{3.10}$$

Again m_g appears as the separation constant between the space-time and internal parts of (3.5) and $m_p{}^2$ that of (3.6). The mass separation constants are analogous to the angular momentum separation constant that arises when the time-independent Schrödinger equation with a central potential is separated into a radial part and an angular part or to the energy separation constant that arises when a time-dependent Schrödinger equation is separated into a time-dependent part and a time-independent part.

Another possibility for achieving a symmetry between the space-time parts and the internal parts of the right sides of (3.3) and (3.4) is to generalize them to the following product forms:

$$i\gamma_5 U_p(x)\psi(x)\xi^a(z) \to i\gamma_5 U_p(x)\psi(x)\tau(z)\xi^a(z) \tag{3.11}$$

$$\mu_p \bar{\chi}(x)\gamma_5 \chi(x)\tau(z) \to \mu_p \mu_0 \bar{\chi}(x)\gamma_5 \chi(x)\zeta_c(z)\zeta^c(z) \tag{3.12}$$

The equations obtained by combining (3.3) with (3.11) and (3.4) with (3.12) are, however, generally not separable. The convenient mass separation constants generally do not exist and these equations may be difficult to solve. As an example to further illustrate this aspect, let us consider the generalization of the following one-dimensional Schrödinger equation:

$$\frac{\partial^2}{\partial x_1{}^2}\psi_1(x_1) + E\psi_1(x_1) = V_1(x_1)\psi_1(x_1) \tag{3.13}$$

to a two-dimensional one. First, multiply (3.13) by $\psi_2(x_2)$ from the right and replace $\partial^2/\partial x_1{}^2$ by $\partial^2/\partial x_1{}^2 + \partial^2/\partial x_2{}^2$ to obtain

$$\left(\frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}\right)\psi_1(x_1)\psi_2(x_2) + E\psi_1(x_1)\psi_2(x_2) = V_1(x_1)\psi_1(x_1)\psi_2(x_2) \tag{3.14}$$

If we let x_1 represent the space time coordinates x and x_2 the internal coordinates z in (3.1) and (3.2), the transition from (3.13) to (3.14) is analogous to the transition from (3.1) and (3.2) to (3.3) and (3.4). Now the interaction term on the right side of (3.14) is not symmetric with respect to x_1 and x_2. One possibility is to generalize $V_1(x_1)$ to the symmetric product form $V_1(x_1)V_2(x_2)$. which leads to a Schrödinger equation generally not separable into x_1 and x_2 parts. If $V_1(x_1) = x_1{}^2$ and $V_2(x_2) = x_2{}^2$, then the interaction potential is $V_1(x_1)V_2(x_2) = x_1{}^2 x_2{}^2$, a form of little interest in atomic

physics. Such a generalization to a product form is analogous to the generalizations to the product forms given in (3.11) and (3.12). Another possibility is to generalize $V_1(x_1)$ to the symmetric sum form $V_1(x_1) + V_2(x_2)$, which leads to a Schrödinger equation separable in x_1 and x_2. In particular, for $V_1(x_1) + V_2(x_2) = x_1^2 + x_2^2$, two harmonic oscillator equations, one in x_1 and the other in x_2, are obtained. This last generalization to a sum form is analogous to the generalization of (3.3) and (3.4) to (3.5) and (3.6), and this sum type of generalization will be adopted.

So far in this section, the interaction function for the two-quark system has been assumed to be a psuedoscalar singlet function $U_p(x)\tau(z)$ generated by one of the quark triplet functions, $\chi(x)\xi^a(z)$. In principle, however, the singlet function $\tau(z)$ can be associated with, instead of the psuedoscalar function $U_p(x)$, also a scalar, vector, psuedovector, or tensor function in space-time. In addition to the singlet function $\tau(z)$, there is a nonet function $\omega_b{}^a(z)$ which can be associated with $U_p(x)$ or any one of the other four mentioned interaction functions in space-time to form an interaction function on the same footing as $U_p(x)\tau(z)$. For instance, if the interaction function is a vector nonet function, denoted by $U_\mu(x)\omega_b{}^a(z)$, (3.5) and (3.6) become

$$i\gamma^\mu\partial_\mu\psi(x)\xi^a(z) + \partial_b{}^a\psi(x)\xi^b(z) = \gamma^\mu U_\mu(x)\psi(x)\xi^a(z) + \omega_b{}^a(z)\psi(x)\xi^b(z) \quad (3.15)$$

$$(\Box - \Diamond)U_\mu(x)\omega_b{}^a(z) = \mu_v\chi(x)\gamma_\mu\bar{\chi}(x)\omega_b{}^a(z) - \mu_n\zeta_b(z)\zeta^a(z)U_\mu(x) \quad (3.16)$$

respectively, where μ_v is a vector interaction parameter and μ_n a nonet interaction parameter. Upon separation, these two equations become

$$i\gamma^\mu\partial_\mu\psi(x) - m_{gv}\psi(x) = \gamma^\mu U_\mu(x)\psi(x) \quad (3.17)$$

$$\partial_b{}^a\xi^b(z) + m_{gv}\xi^a(z) = \omega_b{}^a(z)\xi^b(z) \quad (3.18)$$

$$(\Box - m_v{}^2)U_\mu(x) = \mu_v\chi(x)\gamma_\mu\bar{\chi}(x) \quad (3.19)$$

$$(\Diamond - m_v{}^2)\omega_b{}^a(z) = \mu_n\zeta_b(z)\zeta^a(z) \quad (3.20)$$

where m_{gv} and $m_v{}^2$ are two new mass separation constants.

In general terms, each of the two internal interaction functions, the singlet interaction function $\tau(z)$ and the nonet interaction function $\omega_b{}^a(z)$, can be associated with each of the five interaction functions in space-time, namely, the scalar, psuedoscalar, vector, psuedovector, and tensor interaction functions. Therefore, one has in principle $2 \times 5 = 10$ different interaction functions to choose from. Furthermore, the general interaction function for the two-quark system can in principle consist of a linear combination of the 10 different interaction functions mentioned.

4. *Symmetry-Breaking Interactions*

Equations (3.5) and (3.6) are written in tensor form both in space-time and in the internal space. They are therefore invariant under Lorentz transformations and unitary transformations in the internal space. Further, since the two inter-

acting quarks, represented by $\psi(x)\xi^a(z)$ and $\chi(x)\zeta^a(z)$ respectively, are by themselves identical, (3.5) and (3.6) also hold when $\psi(x)$ is replaced by $\chi(x)$ and vice versa and simultaneously $\xi^a(z)$ is replaced by $\zeta^a(z)$ and vice versa. Thus, the mass separation constants in (3.7)–(3.10) can become invariants under unitary transformations in the internal space. The above discussion obviously also holds for (3.15), (3.16), and (3.17)–(3.20) and for the case with a more general interaction function.

Such a conclusion appears to disagree with the rather successful proposal that the masses of the members of an SU_3 multiplet consist of a term invariant under SU_3 transformations and a term transforming like the eighth component of an SU_3 octet. Further, there are smaller electromagnetic corrections to these masses. Taking these considerations into account, (3.5) is further generalized to include two SU_3 symmetry-breaking interaction terms:

$$i\gamma^\mu\partial_\mu\psi(x)\xi^a(z) + \partial_b{}^a\psi(x)\xi^b(z) = i\gamma_5 U_p(x)\psi(x)\xi^a(z) + [\tau(z) + G_m(z)(\lambda_8)_b{}^a + G_{em}(z)Q_b{}^a]\psi(x)\xi^b(z) \tag{4.1}$$

where

$$2Q = \lambda_3 + \lambda_8/\sqrt{3} \tag{4.2}$$

and the λ's are two of the Gell-Mann matrices. $G_m(z)$ is supposed to represent a semistrong interaction and $G_{em}(z)$ an electromagnetic interaction in the internal space. Since the G_m and G_{em} terms are assumed to be corrections to the singlet term $\tau(z)$ in (4.1), they are assumed to obey equations similar to (3.10):

$$(\Diamond - m_8{}^2)G_m(z) = \mu_8\zeta_a(z)\zeta^a(z) \tag{4.3}$$

$$(\Diamond - m_Q{}^2)G_{em}(z) = \mu_Q\zeta_a(z)\zeta^a(z) \tag{4.4}$$

Here, μ_8 is a semistrong interaction parameter and μ_Q an electromagnetic interaction parameter in the internal space. Guided by the Gell-Mann–Okubo formula (Okubo, 1962), m_8 is assumed to be equal to m_p and the associated $G_m(z)$ is denoted by $G_{mp}(z)$. Equations (3.6), (3.8), and (3.10) now become

$$(\Box - \Diamond)U_p(x)(\tau_{p8})_b{}^a(z) = \mu_p\bar{\chi}(x)\gamma_5\chi(x)(\tau_{p8})_b{}^a(z) - (\mu_0 + \mu_8(\lambda_8)_b{}^a)\zeta_c(z)\zeta^c(z)U_p(x) \tag{4.5}$$

$$\partial_b{}^a\xi^b(z) + m_g\xi^a(z) = (\tau_{p8})_b{}^a(z)\xi^b(z) \tag{4.6}$$

$$(\Diamond - m_p{}^2)(\tau_{p8})_b{}^a(z) = (\mu_0 + \mu_8(\lambda_8)_b{}^a)\zeta_c(z)\zeta^c(z) \tag{4.7}$$

respectively, where

$$(\tau_{p8})_b{}^a(z) = \tau(z) + G_{mp}(z)(\lambda_8)_b{}^a \tag{4.8}$$

The internal electromagnetic interaction $G_{em}(z)$ is naturally associated with the electromagnetic interaction in space-time. Therefore, a corresponding

space-time term for the electromagnetic interaction, $\gamma^{\mu}A_{\mu}(x)\xi^{a}(z)$, is added to the right side of (4.1), where

$$\Box A_{\mu}(x) = \mu_{\alpha}\bar{\chi}(x)\gamma_{\mu}\chi(x) \tag{4.9}$$

and μ_{α} is of the order of the fine structure constant $\approx 137^{-1}$. Since the photon mass is zero, m_Q in (4.4) is therefore also put equal to zero so that (4.4) becomes

$$\diamondsuit G_{em}(z) = \mu_Q \zeta_a(z)\zeta^{a}(z) \tag{4.10}$$

When the interaction function is $U_p(x)(\tau_{p8})_b{}^a(z)$, $U_p(x)$ is coupled to $(\tau_{p8})_b{}^a(z)$ via the mass separation constant $m_p{}^2$. If the interaction function is $U_{\mu}(x)(\omega_{v8})_b{}^a(z)$, where

$$(\omega_{v8})_b{}^a(z) = \omega_b{}^a(z) + G_{mv}(z)(\lambda_8)_b{}^a \tag{4.11}$$

$U_{\mu}(x)$ is coupled to $(\omega_{v8})_b{}^a(z)$ via the mass separation constant $m_v{}^2$. Here, $G_m(z) \to G_{mv}(z)$ and $m_8 \to m_v$ in (4.3). In the case of the electromagnetic interaction, however, such a mass separation constant vanishes. The internal electromagnetic interaction function $Q_b{}^a G_{em}(z)$ is decoupled from the electromagnetic interaction function in space-time, the photon function $A_{\mu}(x)$. This may be associated with the fact that the classical Hamiltonian (2.19) can be generalized to include $A_{\mu}(x)$, but not any other type of interaction, at least up to now, so that

$$[p_{\mu} - A_{\mu}(x)]\,[p^{\mu} - A^{\mu}(x)] - m^2 = 0 \tag{4.12}$$

Following the prescriptions between (2.19) and (2.21), we obtain

$$[i\partial_{\mu} - A_{\mu}(x)]\,[i\partial^{\mu} - A^{\mu}(x)] - \varphi(x)\tau(z) = 0 \tag{4.13}$$

which replaces (2.21). The remark between (2.22) and (2.23) that all the observable quantities in the classical Hamiltonian (2.19) were replaced by operators in (2.21) also holds for the transition from (4.12) to (4.13) since $A_{\mu}(x)$ is not an observable quantity. Separation of (4.13) gives (2.24) and

$$[i\partial_{\mu} - A_{\mu}(x)]\,[i\partial^{\mu} - A^{\mu}(x)] - m_m{}^2\varphi(x) = 0 \tag{4.14}$$

which replaces (2.23).

Taking the SU_3 symmetry-breaking psuedoscalar singlet and vector nonet interaction functions and the photon function, shown between (4.8) and (4.9), into account, (4.1) is further generalized to

$$\begin{aligned} i\gamma^{\mu}\partial_{\mu}\psi(x)\xi^{a}(z) + \partial_b{}^a\psi(x)\xi^{b}(x) = {} & [i\gamma_5 U_p(x) + \gamma^{\mu}U_{\mu}(x) + \gamma^{\mu}A_{\mu}(x)]\psi(x)\xi^{a}(z) \\ & + [(\tau_{p8})_b{}^a(z) + (\omega_{v8})_b{}^a(z) + G_{em}(z)Q_b{}^a]\psi(x)\xi^{b}(z) \end{aligned} \tag{4.15}$$

Just like (3.6) was generalized to (4.5), (3.16) is similarly generalized to

$$\begin{aligned} (\Box - \diamondsuit)U_{\mu}(x)(\omega_{v8})_b{}^a(z) = {} & \mu_v\bar{\chi}(x)\gamma_{\mu}\chi(x)(\omega_{v8})_b{}^a(z) - [\mu_n\zeta_b{}^a(z)\zeta^{a}(z) \\ & + \mu_8\zeta^{c}(z)\zeta_c(z)(\lambda_8)_b{}^a]\,U_{\mu}(x) \end{aligned} \tag{4.16}$$

(4.14) together with (4.5) and (4.16) form a set of equations describing the influence of a quark triplet with the supposedly known wave function $\chi(x)\zeta^a(z)$ upon another with the wave function $\psi(x)\xi^a(z)$. Equations (4.15) and (4.16) can, like (4.5), be separated. As remarked at the end of Section 3, it is in principle possible to suitably include scalar, psuedovector, and tensor interaction functions in (4.15) in addition to the psuedoscalar and vector interaction functions present in it.

It may be noted that the vector field $U_\mu(x)$ associated with the internal none interaction function $(\omega_{v8})_b{}^a(z)$ in (4.15) and (4.16) is not a generalized Yang-Mills (Yang and Mills, 1954) type of field. In (4.15), the orientations of the z^a axes in the internal space remain unchanged for different space-time points.

5. *Generalized Bethe-Salpeter Equation*

In Sections 3 and 4, the interaction between two quarks was described assuming that the wave function of one of the quarks, $\chi(x)\zeta^a(z)$, was known. This is generally not true. In quantum field theory, the interaction between two fermions is described by the Bethe-Salpeter equation (Nambu, 1950; Salpeter and Bethe, 1951). In this investigation only the so-called ladder approximation of this equation, namely,

$$(i\gamma_{\mathrm{I}}{}^\mu\partial_{\mu\mathrm{I}} - m_{\mathrm{I}})(i\gamma_{\mathrm{II}}{}^\mu\partial_{\mu\mathrm{II}} - m_{\mathrm{II}})\Psi_{2q}(x_{\mathrm{I}}, x_{\mathrm{II}}) = G_{pv}(x_{\mathrm{I}}, x_{\mathrm{II}})\Psi_{2q}(x_{\mathrm{I}}, x_{\mathrm{II}}) \qquad (5.1)$$

will be considered. Here, I and II refer to particles I and II, respectively. $\Psi_{2q}(x_{\mathrm{I}}, x_{\mathrm{II}})$ is a 16-component wave function in space-time and $G_{pv}(x_{\mathrm{I}}, x_{\mathrm{II}})$ is an interaction function corresponding to a one-particle exchange between the two interacting fermions.

If a psuedoscalar particle, a vector particle, and a photon are exchanged, as was indicated in (4.15), G_{pv} takes the form

$$G_{pv}(x_{\mathrm{I}}, x_{\mathrm{II}}) = G_{pv}(|x_{\mathrm{I}} - x_{\mathrm{II}}|) = i^2\gamma_{5\mathrm{I}}\gamma_{5\mathrm{II}}G_{pm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)$$
$$+ g_{\mu\nu}\gamma_{\mathrm{I}}{}^\mu\gamma_{\mathrm{II}}{}^\nu[G_{vm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|) + G_{ph}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)] \qquad (5.2)$$

where $G_{pm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)$ is proportional to a relativistic generalization of the Yukawa potential with a mass m_p and an interaction parameter μ_p, $G_{vm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)$ to a similar potential with a mass m_v and an interaction parameter μ_v, $G_{ph}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)$ to $\alpha_f|x_{\mathrm{I}} - x_{\mathrm{II}}|^{-2}$ where $\alpha_f \sim 137^{-1}$, and $g_{\mu\nu} = \delta_{\mu\nu}$ for $\nu = 0$ and $g_{\mu\nu} = -\delta_{\mu\nu}$ otherwise.

The generalization of (5.1) to describe the interaction of two quarks begins by following a prescription similar to that intervening between (3.2) and (3.3). Equation (5.1) is multiplied by an internal two-quark function $\Xi^{bd}(z_{\mathrm{I}}, z_{\mathrm{II}})$ from the right, m_{I} is replaced by $-\partial^a_{b\mathrm{I}}$, m_{II} by $-\partial^c_{d\mathrm{II}}$, $\partial_{\mu\mathrm{I}}$ is multiplied by $\delta_b{}^a$, $\partial_{\mu\mathrm{II}}$ by $\delta_d{}^c$, and G_{pv} by $\delta_b{}^a\delta_d{}^c$. The right side of the resulting equation is, like that of (3.3), not symmetric with respect to the space-time and the internal

interaction functions. The equation is therefore completed in a way similar to that in which (3.3) was completed to become (4.15). The result is

$$
\begin{aligned}
&(i\gamma_{\mathrm{I}}{}^{\mu}\partial_{\mu\mathrm{I}}\delta_b{}^a + \partial^a_{b\mathrm{I}})(i\gamma_{\mathrm{II}}{}^{\mu}\partial_{\mu\mathrm{II}}\delta_d{}^c + \partial^c_{d\mathrm{II}})\Psi_{2g}(x_{\mathrm{I}}, x_{\mathrm{II}})\Xi^{bd}(z_{\mathrm{I}}, z_{\mathrm{II}}) \\
&= G_{pv}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)\Psi_{2g}(x_{\mathrm{I}}, x_{\mathrm{II}})\Xi^{ac}(z_{\mathrm{I}}, z_{\mathrm{II}}) \\
&\quad + \{\tau'(|z_{\mathrm{I}} - z_{\mathrm{II}}|)\delta_b{}^a\delta_d{}^c + \omega'(|z_{\mathrm{I}} - z_{\mathrm{II}}|)(\lambda_p)_b{}^a(\lambda_p)_d{}^c \\
&\quad + [G'_{mp}(|z_{\mathrm{I}} - z_{\mathrm{II}}|) + G'_{mv}(|z_{\mathrm{I}} - z_{\mathrm{II}}|)]\,[(\lambda_8)_b{}^a\delta_d{}^c + \delta_b{}^a(\lambda_8)_d{}^c] \\
&\quad + G_{em}(|z_{\mathrm{I}} - z_{\mathrm{II}}|)(Q_b{}^a\delta_d{}^c + \delta_b{}^aQ_d{}^c)\}\Psi_{2g}(x_{\mathrm{I}}, x_{\mathrm{II}})\Xi^{bd}(z_{\mathrm{I}}, z_{\mathrm{II}}) \qquad (5.3)
\end{aligned}
$$

Here, z_{I} and z_{II} denote the positions of the two-quark triplets or quarks in the internal space. The internal interaction functions are assumed to depend upon $|z_{\mathrm{I}} - z_{\mathrm{II}}|$ only in analogy with the case of the space-time interaction functions given in (5.2) and satisfy

$$(\Diamond - m_p{}^2)\tau'(|z_{\mathrm{I}} - z_{\mathrm{II}}|) = \mu_0\delta(z_{\mathrm{I}} - z_{\mathrm{II}}) \qquad (5.4)$$

$$(\Diamond - m_v{}^2)\omega'(|z_{\mathrm{I}} - z_{\mathrm{II}}|) = \mu_n\delta(z_{\mathrm{I}} - z_{\mathrm{II}}) \qquad (5.5)$$

$$(\Diamond - m_p{}^2)G'_{mp}(|z_{\mathrm{I}} - z_{\mathrm{II}}|) = \mu_8\delta(z_{\mathrm{I}} - z_{\mathrm{II}}) \qquad (5.6)$$

$$(\Diamond - m_v{}^2)G'_{mv}(|z_{\mathrm{I}} - z_{\mathrm{II}}|) = \mu_8\delta(z_{\mathrm{I}} - z_{\mathrm{II}}) \qquad (5.7)$$

$$G'_{em}(|z_{\mathrm{I}} - z_{\mathrm{II}}|) = \mu_{\mathrm{Q}}\delta(z_{\mathrm{I}} - z_{\mathrm{II}}) \qquad (5.8)$$

Here, $\Diamond$ is to be interpreted as operating on $z_{\mathrm{I}} - z_{\mathrm{II}}$ instead of on z as was the case in the previous sections. Obviously, (5.4) has its origin in (3.10), (5.5) in (3.20), (5.6) in (4.3) with $m_8 \to m_p$, (5.7) in (4.3) with $m_8 \to m_v$, and (5.8) in (4.10). The two λ_8 terms and the two Q terms in (5.3) instead of one each in (4.15) are introduced to assure symmetry of (5.3) with respect to the interchange of the two quarks. Further, λ_s are the Gell-Mann matrices with s running from 0 to 8. λ_0 is defined here as

$$\lambda_0 = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (5.9)$$

The indices a, b, c, and d each runs from 1 to 3.

If one of the two quarks is replaced by an antiquark, (5.3) is to be modified in the following way. The space-time part of (5.3) is modified by multiplying (5.3) by $\gamma_1\gamma_3$ from the right (Goldstein, 1953) and by replacing $\Psi_{2q}\gamma_1\gamma_3$ by a quark-antiquark space-time wave function $\Psi_{qa}(x_{\mathrm{I}},x_{\mathrm{II}})$. The internal part of (5.3) is modifed by changing the contravariant indices b and d to covariant ones and the covariant indices d to contravariant ones. In this way, the generalized Bethe-Salpeter equation in the ladder approximation for a quark-antiquark

system reads

$$
\begin{aligned}
&(i\gamma^{\mu}\partial_{\mu\mathrm{I}}\delta_b^{\ a} + \partial_{b\mathrm{I}}^{a})\Psi_{qa}(x_{\mathrm{I}},x_{\mathrm{II}})\Xi_d^{\ b}(z_{\mathrm{I}},z_{\mathrm{II}})(i\gamma^{\mu}\partial_{\mu\mathrm{II}}\delta_c^{\ d} + \partial_{c\mathrm{II}}^{d}) \\
&= i^2 G_{pm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)\gamma_5\Psi_{qa}(x_{\mathrm{I}},x_{\mathrm{II}})\gamma_5\Xi_c^{\ a}(z_{\mathrm{I}},z_{\mathrm{II}}) \\
&\quad + [G_{ph}(|x_{\mathrm{I}} - x_{\mathrm{II}}|) + G_{vm}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)]\gamma_{\mu}\Psi_{qa}(x_{\mathrm{I}},x_{\mathrm{II}})\gamma^{\mu}\Xi_c^{\ a}(z_{\mathrm{I}},z_{\mathrm{II}}) \\
&\quad + \{\tau'(|z_{\mathrm{I}} - z_{\mathrm{II}}|)\delta_b^{\ a}\delta_d^{\ c} + \omega'(|z_{\mathrm{I}} - z_{\mathrm{II}}|)(\lambda_s)_b^{\ a}(\lambda_s)_d^{\ c} \\
&\quad + [G'_{mp}(|z_{\mathrm{I}} - z_{\mathrm{II}}|) + G'_{mv}(|z_{\mathrm{I}} - z_{\mathrm{II}}|)][(\lambda_8)_b^{\ a}\delta_d^{\ c} + \delta_b^{\ a}(\lambda_8)_d^{\ c}] \\
&\quad + G'_{em}(|z_{\mathrm{I}} - z_{\mathrm{II}}|)(Q_b^{\ a}\delta_d^{\ c} + \delta_b^{\ a}Q_d^{\ c})\}\Psi_{qa}(x_{\mathrm{I}},x_{\mathrm{II}})\Xi_d^{\ b}(z_{\mathrm{I}},z_{\mathrm{II}})
\end{aligned}
\tag{5.10}
$$

As remarked at the ends of Sections 3 and 4, (5.10) can be generalized to include scalar, psuedovector, and tensor interactions in space-time. Such an equation has been treated in an accompanying paper (Hoh, 1975).

6. *Transformation to Spherical Coordinates in M_3*

It is useful to introduce spherical coordinates and associated spherical harmonics in the internal space that are analogous to the ordinary spherical coordinates and spherical harmonics in three-dimensional real space.

Following Bég and Ruegg (1965) closely, the following transformation is made:

$$
\begin{aligned}
z^1 &= r\sin\vartheta\cos\xi\, e^{i\varphi_1} \\
z^2 &= r\sin\vartheta\sin\xi\, e^{i\varphi_2} \\
z^3 &= r\cos\vartheta\, e^{i\varphi_3}
\end{aligned}
\tag{6.1}
$$

where

$$
0 \leqslant \vartheta, \xi \leqslant \pi/2, \quad 0 \leqslant \varphi_1, \varphi_2, \varphi_3 \leqslant 2\pi \tag{6.2}
$$

Taking the complex conjugate of (6.1), one obtains

$$
\begin{aligned}
z_1 &= r\sin\vartheta\cos\xi\, e^{-i\varphi_1} \\
z_2 &= r\sin\vartheta\sin\xi\, e^{-i\varphi_2} \\
z_3 &= r\cos\vartheta\, e^{-i\varphi_3}
\end{aligned}
\tag{6.3}
$$

by means of (6.1) and (6.2), one can write

$$
\partial_1 \equiv \frac{\partial}{\partial z^1} = \frac{1}{2}e^{-i\varphi_1}\left(\sin\vartheta\cos\xi\frac{\partial}{\partial r} + \cos\vartheta\cos\xi\frac{1}{r}\frac{\partial}{\partial\vartheta} - \frac{\sin\xi}{r\sin\vartheta}\frac{\partial}{\partial\xi} - \frac{i}{r\sin\vartheta\cos\xi}\frac{\partial}{\partial\varphi_1}\right)
$$

$$
\partial_2 \equiv \frac{\partial}{\partial z^2} = \frac{1}{2}e^{-i\varphi_2}\left(\sin\vartheta\sin\xi\frac{\partial}{\partial r} + \cos\vartheta\sin\xi\frac{1}{r}\frac{\partial}{\partial\vartheta} + \frac{\cos\xi}{r\sin\vartheta}\frac{\partial}{\partial\xi} - \frac{i}{r\sin\vartheta\sin\xi}\frac{\partial}{\partial\varphi_2}\right)
$$

$$
\partial_3 \equiv \frac{\partial}{\partial z^3} = \frac{1}{2}e^{-i\varphi_3}\left(\cos\vartheta\frac{\partial}{\partial r} - \sin\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta}\frac{i}{r\cos\vartheta}\frac{\partial}{\partial\varphi_3}\right)
$$

$$
\partial^1 = \partial_1^*, \quad \partial^2 = \partial_2^*, \quad \partial^3 = \partial_3^* \tag{6.4}
$$

A volume element in the internal space M_3 can be written as

$$d^6z \equiv dz_1\, dz^1\, dz_2\, dz^2\, dz_3\, dz^3 = |J_6|\, dr\, d\vartheta\, d\varphi_1\, d\varphi_2\, d\varphi_3 \tag{6.5}$$

where the absolute value of the Jacobian J_6 is given by

$$|J_6| = 8r^5 \cos\vartheta \sin^3\vartheta \cos\xi \sin\xi \tag{6.6}$$

Bég and Ruegg (1965) defined a set of harmonic functions for the group SU_3 as follows:

$$Y_n^{m_1 m_2 m_3 I}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \frac{1}{\sin\vartheta} e^{i(m_1\varphi_1 + m_2\varphi_2 + m_3\varphi_3)}$$

$$\times\, d^{(n+1)/2}_{(m_3+2I+1)/2,(m_3-2I-1)/2}(2\vartheta) d^I_{(m_1+m_2)/2,(m_1-m_2)/2}(2\xi) \tag{6.7}$$

They also pointed out that these functions (6.7) form a complete orthogonal set in the intervals specified in (6.2) using $|J_6|/8r^5$ as the density function. Here, the set of harmonic functions is normalized:

$$Y_{nI}^{m_1 m_2 m_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = e^{i\delta}\left[\frac{(2I+1)(n+2)}{2\pi^3}\right]^{1/2} Y_n^{m_1 m_2 m_3} \tag{6.8}$$

Where δ is a phase angle not yet determined and may depend upon m_1, m_2, m_3, I, and n. Using the density function $|J_6|/8r^5$ and observing (6.2), one can verify the following orthonormality relation:

$$\int Y_{n'I'}^{m_1' m_2' m_3'} {}^{*} Y_{nI}^{m_1 m_2 m_3} \frac{1}{8r^5 dr} d^6z = \delta_{m_1' m_1} \delta_{m_2' m_2} \delta_{m_3' m_3} \delta_{n' n} \delta_{I' I} \tag{6.9}$$

A function of z_a and z^a, with $a = 1$, 2, and 3, can be expanded as follows:

$$F(z_a, z^a) = f(r, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \sum_{n=0}^{\infty} \sum_{2I=0}^{n} \sum_{m_3=-(n-2I)}^{n-2I} \sum_{m_2=-2I}^{2I} \sum_{m_1=-(2I-|m_2|)}^{2I-|m_2|}$$

$$f_{nI}^{m_1 m_2 m_3}(r) Y_{nI}^{m_1 m_2 m_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) \tag{6.10}$$

One can also show the following completeness relation:

$$\sum_{n=0}^{\infty} \sum_{2I=0}^{n} \sum_{m_3=-(n-2I)}^{n-2I} \sum_{m_2=-2I}^{2I} \sum_{m_1=-(2I-|m_2|)}^{2I-|m_2|} Y_{nI}^{m_1 m_2 m_3} {}^{*}(\vartheta', \xi', \varphi_1', \varphi_2', \varphi_3')$$

$$\times\, Y_{nI}^{m_1 m_2 m_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = \frac{8r^5}{|J_6|} \delta(\vartheta' - \vartheta)\delta(\xi' - \xi)\delta(\varphi_1' - \varphi_1)\delta(\varphi_2' - \varphi_2)\delta(\varphi_3' - \varphi_3) \tag{6.11}$$

In accordance with Bég and Ruegg (1965), the following identifications with quark theory are made:

$$p + q = n, \quad p - q = m_1 + m_2 + m_3, \quad Y = \tfrac{1}{3}(-2m_1 + m_2 + m_3), \quad I_3 = \tfrac{1}{2}(m_1 - m_2) \tag{6.12}$$

From this equation one obtains

$$m_1 = \tfrac{1}{3}(p-q) + \tfrac{1}{2}Y + I_3, \quad m_2 = \tfrac{1}{3}(p-q) + \tfrac{1}{2}Y - I_3, \quad m_3 = \tfrac{1}{3}(p-q) - Y \tag{6.13}$$

(6.8) and (6.7) can now be rewritten as follows:

$$Y^{pq}_{YII_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = Y^{m_1 m_2 m_3}_{nI} = e^{i\delta}\left[\frac{(2I+1)(p+q+2)}{2\pi^3}\right]^{1/2} \frac{1}{\sin\vartheta}$$

$$\times\, d^{(p+q+1)/2}_{(p-q-3Y+6I+3)/6,\,(p-q-3Y-6I-3)/6}(2\vartheta)\, d^{I}_{(p-q)/3+Y/2,\,I_3}(2\xi)$$

$$\times \exp\left\{i[\tfrac{1}{3}(p-q) + \tfrac{1}{2}Y + I_3]\varphi_1 + i[\tfrac{1}{3}(p-q) + \tfrac{1}{2}Y - I_3]\varphi_2 + i[\tfrac{1}{3}(p-g) - Y]\varphi_3\right\} \tag{6.14}$$

Similarly, (6.10) can be rewritten as

$$F(z_a, z^a) = f(r, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3)$$

$$= \sum_{p+q=0}^{\infty} \sum_{2I=0}^{p+q} \sum_{Y=-(p+q-2I)}^{p+g-2I} \sum_{I_3=-I}^{I} \sum_{(p-q)/3+Y/2=-I}^{I}$$

$$f(p, g, Y, I, I_3, r)\, Y^{pq}_{YII_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) \tag{6.15}$$

The operator $\partial_a{}^a$ can be written in the following way:

$$\partial_a{}^a \equiv \partial_1{}^1 + \partial_2{}^2 + \partial_3{}^3 = \frac{1}{r^5}\frac{\partial}{\partial r} r^5 \frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_5 \tag{6.16}$$

Where Δ_5 is a second-order operator operating on the angles ϑ, ξ, φ_1, φ_2, and φ_3 only. Bég and Ruegg (1965) have shown that $Y^{pq}_{YII_3}$ is an eigenfunction of Δ_5:

$$\Delta_5 Y^{pq}_{YII_3} = -(p+q)(p+q+4) Y^{pq}_{YII_3} \tag{6.17}$$

7. *One-Quark System*

Returning to the case of free quarks, (2.18) can be solved by assuming that

$$\xi^a(z) = \text{const}\; k^a e^{i(k_b z^b + k^b z_b)} \tag{7.1}$$

so that

$$m_{q0} = k_b k^b = k_1 k^1 + k_2 k^2 + k_3 k^3 \tag{7.2}$$

If a box type of boundary conditions in M_3 is assumed, discrete k_b and k^b values are obtained. The solution (7.1) does not, however, refer to a single quark. The internal wave function of a single quark is obtained by considering

the following expansion of $\xi^a(z)$ for a one-quark system in spherical harmonics:

$$\xi^a(z) = (\xi^1(z), \xi^2(z), \xi^3(z))$$
$$= \Sigma_{p,q,Y,I,I_3} \begin{pmatrix} Q_1(p,q,Y,I,I_3,r) Y^{p+1q}_{Y+\frac{1}{3}I+\frac{1}{2}I_3+\frac{1}{2}} \\ Q_2(p,q,Y,I,I_3,r) Y^{p+1q}_{Y+\frac{1}{3}I+\frac{1}{2}I_3-\frac{1}{2}} \\ Q_3(p,q,Y,I,I_3,r) Y^{p+1q}_{Y-\frac{2}{3}II_3} \end{pmatrix}^T \tag{7.3}$$

where T denotes transposed. Keeping the lowest-order term, associated with $p = g = Y = I = I_3 = 0$, only and putting $Q_1 = Q_2 = Q_3 = Q$, (7.3) becomes

$$\xi^a(z) = Q(0,0,0,0,0,r) \frac{1}{\pi} \sqrt{\frac{6}{\pi}} \begin{pmatrix} \sin\vartheta \cos\xi e^{i\varphi_1} \\ \sin\vartheta \sin\xi e^{i\varphi_2} \\ \cos\vartheta e^{i\varphi_3} \end{pmatrix}^T \tag{7.4}$$

which can be considered as the internal wave function of a single free quark. $\xi^a(z)$ in (7.3) generally represents states more complex than a single quark; for instance, the $p = q = 1$ term may represent a single quark together with a quark-antiquark pair. In obtaining (7.4), (6.14) was used in which δ was put equal to 0 for ξ^2 and ξ^3 but to π for ξ^1. This procedure enables (7.4) to take a form conforming to (6.1) and leads to the cancelation of the angular parts in the derivation of the next equation. Inserting (7.4) into (2.18), and applying (6.4), the angular parts cancel out and one obtains

$$Q'' + (5/r)Q' - (5/r^2)Q + 4m_{q0}Q = 0 \tag{7.5}$$

where $Q' \equiv \partial Q/\partial r$. The solution of (7.5) is

$$Q = \text{const}\ (1/r^2) J_3(2\sqrt{m_{q0}}r) \tag{7.6}$$

where J denotes Bessel's function. m_{q0} can be interpreted as an indeterminate mass of a single free quark.

If a singlet interaction term, $\tau(z)$ in (3.10) is included, (2.18) becomes (3.8). For simplicity, it will be assumed that the source function in (3.10) can be replaced by a point source function $\mu_0\delta(z)$. This assumption is analogous to the assumption in space-time that the source of an interaction function, corresponding to $\tau(z)$, is a point source located at the origin of a three-dimensional real coordinate system in space. Further it will be assumed that the mass separation constant m_p^2 vanishes and that $z \to |z| = r$. Equation (3.10) then becomes

$$\Diamond \tau(r) = \mu_0 \delta(z) \tag{7.7}$$

Making use of (2.22) and (6.16), (3.8) yields

$$\tau(r) = \frac{\mu_0}{128\,\pi^3} \frac{1}{r^2} + \mu_{10} \frac{1}{r^4} + \mu_2 r^2 + \mu_3 \tag{7.8}$$

where μ_{10}, μ_2, and μ_3 are integration constants. In this connection one may note that the equation

$$\partial_a{}^a \tau_0(r) = \nu_0' \delta(z) \tag{7.9}$$

has the solution

$$\tau_0(r) = -\frac{\nu_0'}{32\,\pi^3}\frac{1}{r^4} + \nu_1' \tag{7.10}$$

where ν_1' is an integration constant.

By analogy with $G_{ph}(|x_{\mathrm{I}} - x_{\mathrm{II}}|)$, discussed after (5.2), one may assume that $\tau(r \to \infty) \mapsto \infty$ so that $\mu_2 = 0$.

Equation (7.4) is now substituted into (3.8) and, again, the angular parts cancel out leaving a radial equation like (7.5) with m_{q0} replaced by $m_{q0} - \tau(r)$. Application of Frobenius' method to this radial equation using (7.8) yields no solution when $\mu_{10} \neq 0$. If, however, $\mu_{10} = 0$, one finds that

$$Q = \text{const}\,(1/r^2) J_{\pm\sqrt{9+4\mathrm{mu}}}(2\sqrt{m_q - \mu_3}\,r) \tag{7.11}$$

where $\mathrm{mu} = \mu_0/128\,\pi^3$. Equation (7.11) is a modified form of (7.6) with m_q not determined. Thus, a single quark cannot be confined by a central potential in the internal space of the type (7.8) or (7.10) in the sense that these potentials do not confine the radial internal function Q into a certain region in the internal space and make it vanish exponentially outside that region.

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